Recitation 4. October 1

Focus: bases, and the four fundamental subspaces.

A ***basis*** of a vector space V is a set of vectors $v_1, \dots, v_n$ that I) span and II) are linearly independent. The number of vectors in a basis (which is always the same) is the ***dimension*** of the vector space.

Let A be an $m \times n$ matrix. The ***four fundamental subspaces*** of A are: I) the ***nullspace*** $N(A) \subset \mathbb{R}^n$, II) the ***column space*** $C(A) \subset \mathbb{R}^m$, III) the ***row space*** $C(A^T) \subset \mathbb{R}^n$, and IV) the ***left nullspace*** $N(A^T) \subset \mathbb{R}^m$.

1. Determine if each of the following is a basis for the given vector space. If it is not, add or remove vectors to make it one:

   (a) The set of vectors $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\1\end{bmatrix}$ in $\mathbb{R}^3$.

   (b) The set of vectors $\begin{bmatrix}2\\4\\0\\3\end{bmatrix}, \begin{bmatrix}1\\-1\\2\\0\end{bmatrix}, \begin{bmatrix}6\\0\\0\\0\end{bmatrix}$ in $\mathbb{R}^4$

   (c) The set of vectors $\begin{bmatrix}-3\\1\\2\end{bmatrix}, \begin{bmatrix}1\\-3\\6\end{bmatrix}, \begin{bmatrix}2\\4\\5\end{bmatrix}, \begin{bmatrix}1\\2\\2\end{bmatrix}$ in $\mathbb{R}^3$.

Solution:

2. Recall that for two vector subspaces V, W in $\mathbb{R}^n$, their sum is $V + W = \{v + w \mid v \in V \text{ and } w \in W\}$, and their intersection is $V \cap W = \{v \mid v \text{ is in both } V, W\}$. Let

$$V = \text{Span}\left\{\begin{bmatrix}-1\\0\\1\\4\end{bmatrix}, \begin{bmatrix}2\\6\\0\\-3\end{bmatrix}, \begin{bmatrix}1\\12\\2\\0\end{bmatrix}\right\} \qquad W = \text{Span}\left\{\begin{bmatrix}0\\-3\\1\\-3\end{bmatrix}, \begin{bmatrix}-1\\6\\3\\9\end{bmatrix}\right\}.$$

   Find a basis for $V + W$ and $V \cap W$.

Solution:

3. Use Gauss-Jordan elimination find a basis of each of the four fundamental subspaces of

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 \\ 2 & -1 & 0 & 5 \\ -3 & 2 & -2 & 1 \end{bmatrix}.$$

What is the dimension of each?

Solution:

4. Let B be a square matrix such that $B^T = B^{-1}$. Show that the columns of B are (pairwise) orthogonal and have length 1. A matrix with this property is said to be ***orthogonal***.

Solution: